NOTICE

Experimental and Analytical Study of a Hybrid Isolation System Using Friction Controllable Sliding Bearings

by

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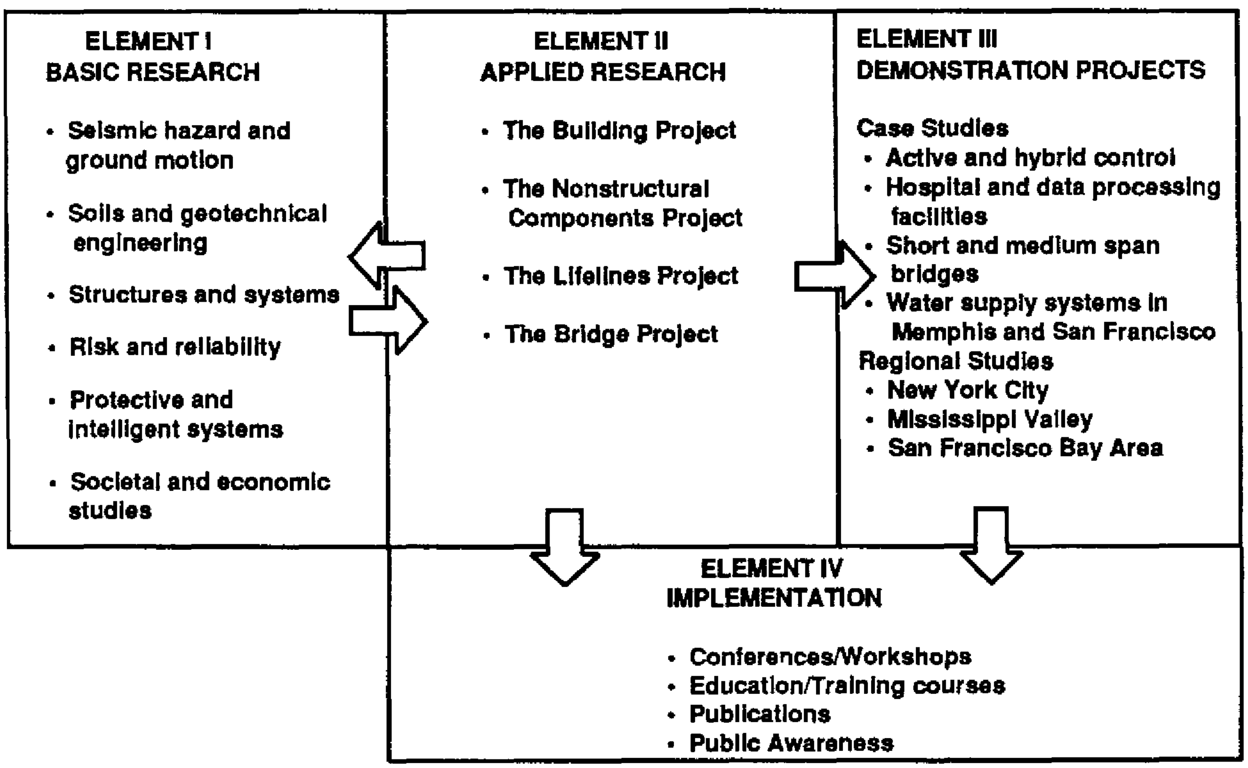
PREFACE

The National Center for Earthquake Engineering Research (NCEER) was established to expand and disseminate knowledge about earthquakes, improve earthquake-resistant design, and implement seismic hazard mitigation procedures to minimize loss of lives and property. The emphasis is on structures in the eastern and central United States and lifelines throughout the country that are found in zones of low, moderate, and high seismicity.

NCEER's research and implementation plan in years six through ten (1991-1996) comprises four interlocked elements, as shown in the figure below. Element I, Basic Research, is carried out to support projects in the Applied Research area. Element II, Applied Research, is the major focus of work for years six through ten. Element III, Demonstration Projects, have been planned to support Applied Research projects, and will be either case studies or regional studies. Element IV, Implementation, will result from activity in the four Applied Research projects, and from Demonstration Projects.



Research in the **Building Project** focuses on the evaluation and retrofit of buildings in regions of moderate seismicity. Emphasis is on lightly reinforced concrete buildings, steel semi-rigid frames, and masonry walls or infills. The research involves small- and medium-scale shake table tests and full-scale component tests at several institutions. In a parallel effort, analytical models and computer programs are being developed to aid in the prediction of the response of these buildings to various types of ground motion.

Two of the short-term products of the **Building Project** will be a monograph on the evaluation of lightly reinforced concrete buildings and a state-of-the-art report on unreinforced masonry.

The **protective and intelligent systems program** constitutes one of the important areas of research in the **Building Project.** Current tasks include the following:

1. Evaluate the performance of full-scale active bracing and active mass dampers already in place in terms of performance, power requirements, maintenance, reliability and cost.
2. Compare passive and active control strategies in terms of structural type, degree of effectiveness, cost and long-term reliability.
3. Perform fundamental studies of hybrid control.
4. Develop and test hybrid control systems.

This report describes the effectiveness of a hybrid isolation system using friction controllable sliding bearings. Two control algorithms, bang-bang control and instantaneous optimal control, are developed to control the friction force. Their effectiveness is demonstrated by shaking table experiments and computer simulation.

The hybrid sliding isolation system using friction controllable bearings is physically developed. Shaking table experiments are performed using a structural model equipped with the hybrid system. These experiments demonstrate the advantage of using the hybrid system over the passive system.

Computer codes to simulate the structural response under passive and hybrid control are developed. The numerically simulated results show good agreement with the experimental results. These results verify that the analytical model developed adequately represents the actual system.

ABSTRACT

This study deals with a hybrid isolation system using friction controllable sliding bearings [1,2]. During earthquakes, this isolation system controls the friction force on the sliding interface between the supported structure and the ground, by adjusting the pressure in a bearing chamber, to confine the sliding displacement within an acceptable range, while keeping the transfer of seismic force to a minimum to obtain the best isolation performance. This is the advantage of the hybrid sliding isolation system that cannot be achieved by the passive sliding system.

Instantaneous optimal control and bang-bang control algorithms are developed for controlling the friction force, since standard control theory is difficult to apply in a straightforward fashion in this case where the control force has a nonlinear feature. The effectiveness of the algorithms in controlling seismic response of a structural model is demonstrated by shaking table experiments and computer simulation.

A hybrid sliding isolation system using friction controllable bearings is physically developed, and shaking table experiments are performed using a rigid structural model equipped with such a hybrid system. The dynamic characteristics of the control system for bearing pressure and sliding friction is identified, and the advantage of the hybrid sliding isolation system over the passive system is demonstrated by experiments.

Computer codes for simulation of structural response under passive or hybrid control are developed. The numerically simulated results show good agreement with the experimental results, verifying that the analytical model developed represents the actual system very well.

Both experimental and analytical studies demonstrate the effectiveness of the hybrid sliding isolation system and suggest its advantageous use in civil structures.

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Contents

List of Figures

List of Tables

SECTION 1

INTRODUCTION

1.1 Background

Seismic base isolation techniques have been used with increasing popularity to protect the structures, together with their occupants, secondary systems and internal equipment, from the damaging effects of earthquakes. The sliding isolation systems, however, have the following problems:

1. A structure supported entirely by sliding bearings experiences forces at the sliding interface that are always bounded by the frictional force, regardless of the intensity and frequency content of the ground excitation. From this view point, the smaller the coefficient, the higher the isolation performance. However, because of unavoidable residual displacement and possibly excessive sliding displacement associated with such systems, particularly those with small coefficients of friction, the purely sliding isolation system is difficult to use.

2. For practice, most of the sliding isolation systems currently in use are equipped with some form of restoring devices [3, 4, 5]. For a passive sliding system with a restoring force, the design of coefficient of friction and stiffness of the restoring device is not a simple problem. If the coefficient of friction is relatively small, then the isolation performance is greatly influenced by the stiffness of the restoring device. As a result, the advantage of the purely sliding isolation system (mentioned above) cannot be realized. For this reason, the coefficient of friction μ is usually designed to the extent (e.g., $\mu = 0.15 \sim 0.20$) that its isolation performance is not unduly influenced by the stiffness of the restoring device [6, 7]. However such a system becomes totally ineffective for the earthquakes with peak ground acceleration less than or equal to μg, in spite of the fact that such small to medium earthquakes occur more often and they can cause damage to sensitive equipment, valuable items and secondary systems inside the building.

3. Since a passive sliding isolation system is usually designed and effective for large earthquakes with long return periods, the system seldom has the opportunity to be activated during the service life of the building. For example, the TASS system (Kawamura et al. 1988) was installed in an office building (called J Building) in Yokohama, Japan four years ago, but has never been activated, although many small earthquakes occurred in that area during this period. Whether or not the sliding system can maintain the initially designed value of coefficient of friction throughout the long period of inactivity remains to be an important practical problem.

1.2 Objectives and Hybrid Isolation System

The objectives of this research are then to physically develop a friction controllable sliding isolation system which can retain the advantages, while eliminating the disadvantages, of the purely sliding isolation system, and thus delivers a fundamentally superior isolation performance which cannot be achieved by the passive sliding isolation system (with a restoring device).

To be more specific, this system can intelligently control the friction force on the sliding interface between the supported structure and the ground so as to confine the sliding displacement in an acceptable range, to reduce the residual displacement, and at the same time, to minimize the transfer of seismic force to the structure. For small to medium earthquakes, the friction force can be controlled to a small level to obtain the best isolation performance. For large earthquakes, the sliding displacement can be confined within an acceptable range by controlling the friction force, while the minimum isolation performance is guaranteed by the maximum friction force of the system. Therefore, this friction controllable sliding isolation system is effective for all intensities of earthquakes, (ranging from small, medium or large), unlike the passive sliding isolation systems which are usually designed for large earthquakes.

Furthermore, the confidence in this system can be more easily established than the passive sliding isolation system for the reason that this system has more opportunities of getting activated even under small earthquakes which occur more frequently.

To achieve these objectives, this research attempts to:

1. propose and physically develop a friction controllable sliding bearing and a hybrid sliding isolation system using such bearings,

2. develop control algorithms for controlling the friction force which has nonlinear characteristics,

3. through experiments and computer simulation analysis, demonstrate the effectiveness of the hybrid isolation system using friction controllable sliding bearings in controlling the seismic responses of both a single-degree-of-freedom structural model and a multi-degree-of-freedom building.

The hybrid isolation system using friction controllable bearings (FCB's) to be investigated is conceptually depicted in Figs. 1.1 and 1.2, respectively with a building and a bridge structure resting on the bearings. Each bearing has a fluid chamber which is connected to a pressure control system composed of a servo valve, an accumulator and a computer. The friction on the interface between the bearing and the ground is controlled by adjusting the fluid pressure in the chamber. The computer calculates an appropriate signal to control the fluid pressure based on the observed structural response, such as response acceleration and sliding displacement, and sends it to the pressure control device as shown in Figs. 1.1 and 1.2.

The idealized section view of the friction controllable sliding bearing is given in Fig. 1.3. The bearing made of steel is of disk shape containing a fluid chamber inside which is sealed by a rubber O-ring around the circular perimeter just inside the sliding interface. A sliding material such as PTFE plate is placed on the sliding surface.

The word "HYBRID" is used for this friction controllable sliding isolation system, since it is a combination of the passive sliding isolation system and the active control device. The system can be a passive sliding isolation system as long as the pressure of the bearing chamber, and thus the friction is kept at a constant value. At the same time, the system can be an active system as long as the pressure is controlled so that the friction is controlled.

In the analytical formulation involving such a hybrid system, the control force does not appear in the equation of motion as an independent term like the force from an actuator. The active control in this case is implemented through the friction term in the equation, and in that sense the system may also be interpreted as "Semi-Active".

This hybrid isolation system has the following general advantages: (a) Changing friction force through controlling pressure requires smaller amounts of energy and power than the corresponding actuator-driven control system, and as a consequence (b), the use of accumulators for the source of energy is possible, thus eliminating the necessity of emergency energy supply system.

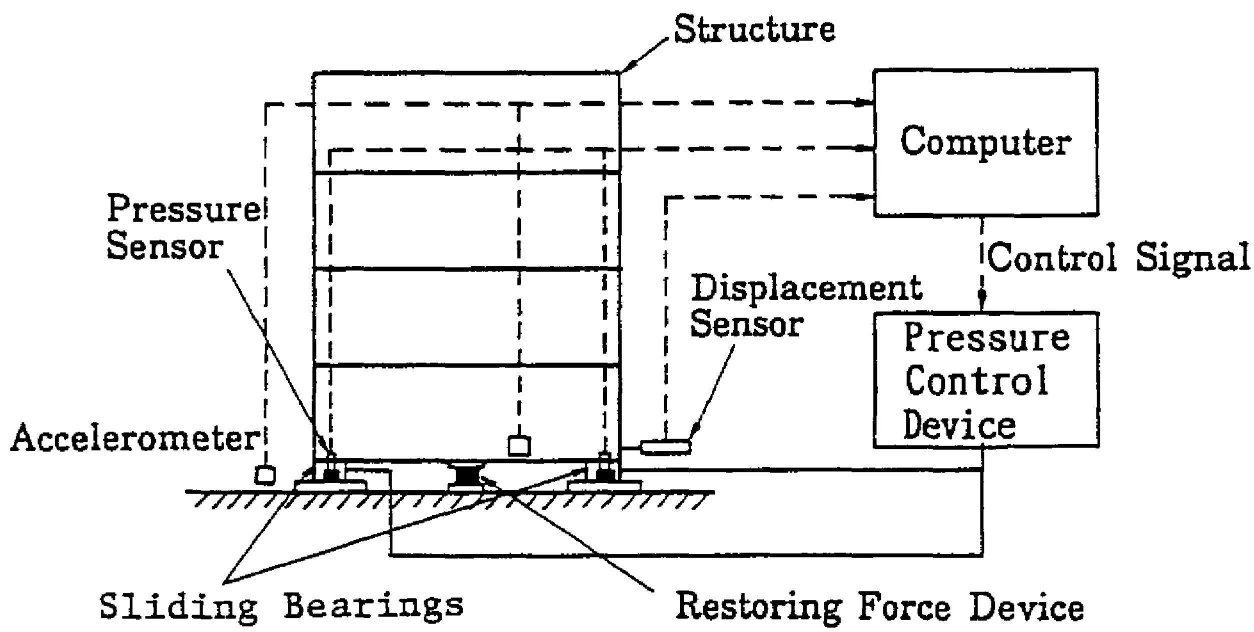

Figure 1.1: Concept of Hybrid Sliding Isolation System for Buildings

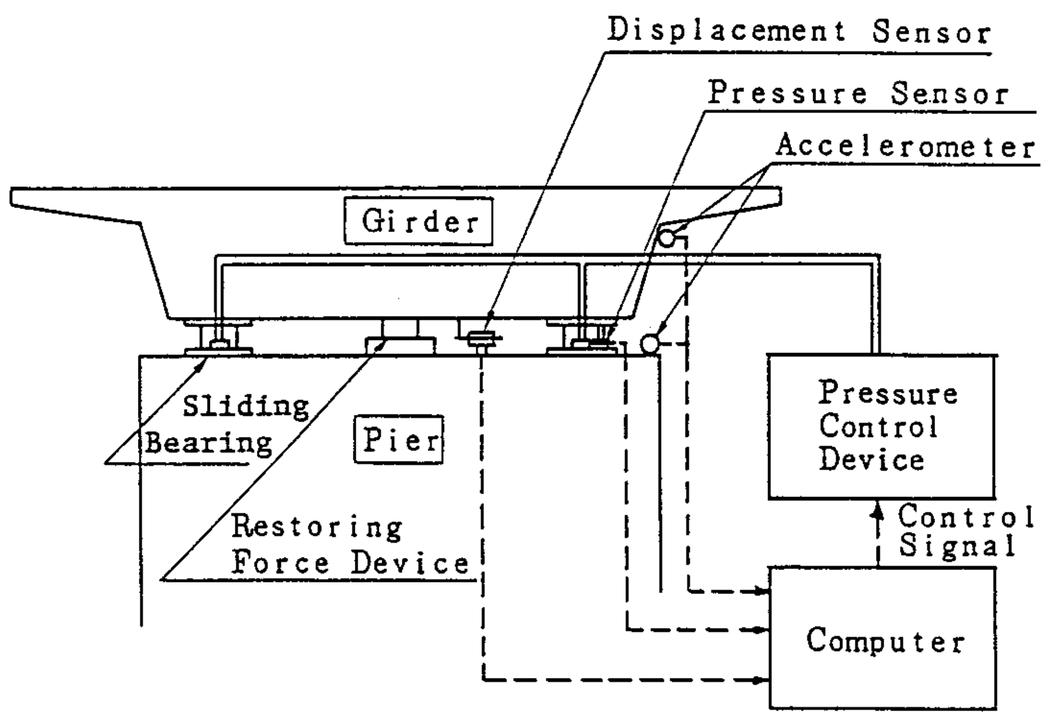

Figure 1.2: Concept of Hybrid Sliding Isolation System for Bridges

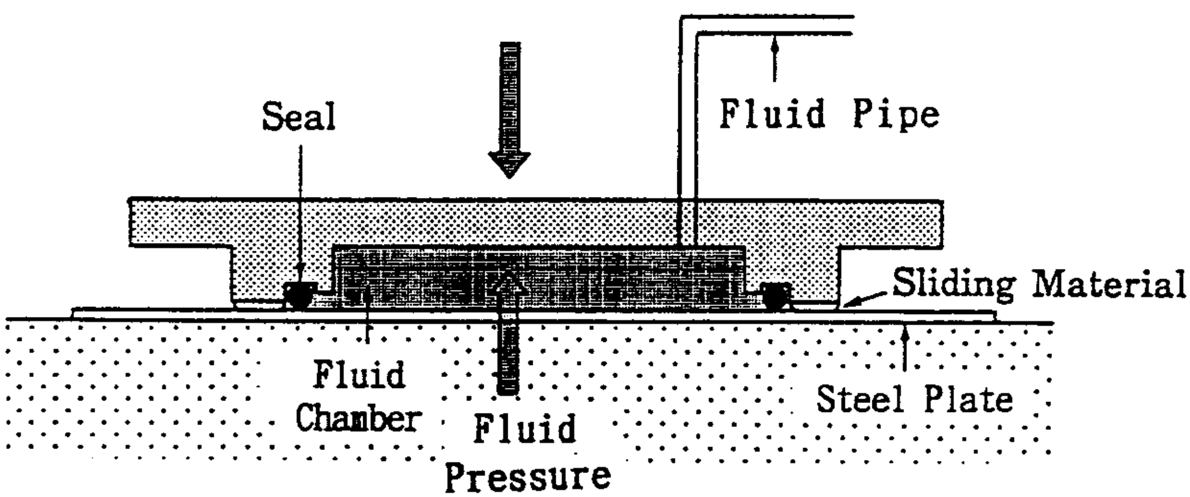

Figure 1.3: Idealized View of Friction Controllable Sliding Bearing

1.3 Outline of the Study

In Section 2, two kinds of control algorithms are newly developed for controlling the friction force in the sliding isolation system, since standard control theory can not be applied in a straightforward fashion in this case where the analytical model of the physical system is nonlinear involving the friction force. These control algorithms are either based on instantaneous optimal control theory or bang-bang control method. As to the instantaneous optimal control, two cases are studied: one takes into account the time delay of the friction control system, while the other does not. These control algorithms, whether based on instantaneous optimal control or bang-bang control, are relatively simple and yet robust for on-line control operations, and they effectively achieve the objectives mentioned above.

In Sections 3 and 4, the experimental study performed on a single-degree-of-freedom structure installed with a hybrid sliding isolation system is described. The sliding bearing and its control device are developed. A single-degree-of-freedom model structure is constructed. Computer codes using the C language for real-time on-line control operations are developed. The characteristics of the hybrid isolation system is identified by experiments. Then, the hybrid isolation system which can control the friction force in the sliding interface between the model structure and the ground (shaking table) is developed. It can act as a passive isolation system by maintaining the friction force at a certain value. Shaking table experiments of the model structure equipped with such a sliding isolation systems are carried out to evaluate its isolation performance under various earthquake excitations with differing levels of intensities. The isolation performance of the hybrid system is compared with that of the passive system, and the advantage of the proposed hybrid sliding isolation system is demonstrated.

In Section 5, related analytical and numerical studies also performed. A computer program for numerical evaluation of the response of the structure under passive or hybrid isolation is developed. The numerical simulation results show a good agreement with the experimental results, confirming the adequacy of the analytical model and the simulation method.

Finally, the conclusion of this report is given in Section 6.

SECTION 2

CONTROL ALGORITHMS

Control of structural response by the proposed hybrid sliding isolation system using friction controllable bearings presents an unique problem in developing control algorithms. The reason is that the control force in this sliding isolation system is the friction force, which depends on the direction of the sliding velocity and thus appears as a nonlinear term in the equation of motion. For controlling such a nonlinear force, standard control theory is difficult to apply, and not much research has been done to derive control algorithms. Fujita et al [8, 9] developed a semi-active seismic isolation system using controllable friction dampers. In this system, the frictional damping force is controlled by changing the pressure between the friction elements using an actuator. Linear optimal control algorithm in modern control theory was applied in a straightforward manner in this study. Since the frictional damping force is a nonlinear force depending on the direction of velocity, the optimal control force required by the linear control theory cannot be fully realized by the friction force, and thus the real "optimal control" can not be implemented. In this respect, Feng and Shinozuka [10, 11] were the first to derive a nonlinear control alorithm on the basis of instantaneous optimal control [12, 13] for the system with such a nonlinear control force. By simulation study, they showed the effectiveness of the control algrithm. In their study, however, nonlinear differential equations are needed to be solved for control force at every control time instant, making real-time on-line control operations rather difficult.

In this study, two types of control algorithms, both of which are implementable in real-time on-line operations, are developed to control the nonlinear friction force in the proposed hybrid isolation system under earthquake loads. They are developed on the basis of the instantaneous optimal control theory and bang-bang control concept. As to the instantaneous optimal control, furthermore, two cases are studied: one with the time delay of the friction control system taken into consideration and the other without.

2.1 Analytical Model

A rigid structure supported by the friction controllable sliding bearings is considered. The motion of the structure can be modeled by a single-degree-of-freedom (SDOF) model as shown in Fig. 2.1. The equations of motion of the structure under earthquake excitation can then be written as follows.

1. Sticking Phase — Phase I

$$\dot{x} = 0, \quad x = const. \tag{2.1}$$

2. Sliding Phase — Phase II

$$\ddot{x} = -\ddot{z} - f \ \mathrm{sgn}(\dot{x}), \tag{2.2}$$

3. Criteria for transition from Phase I to Phase II

$$|\ddot{z}| > f \tag{2.3}$$

4. Criteria for transition from Phase II to Phase I

$$\dot{x} = 0 \tag{2.4}$$

$$|\ddot{x}| < 2f \tag{2.5}$$

where

x: sliding displacement of mass relative to ground

$\ddot{z}$: input earthquake acceleration

μ: coefficient of friction on sliding interface

f: normalized friction force defined as $f = \mu g$

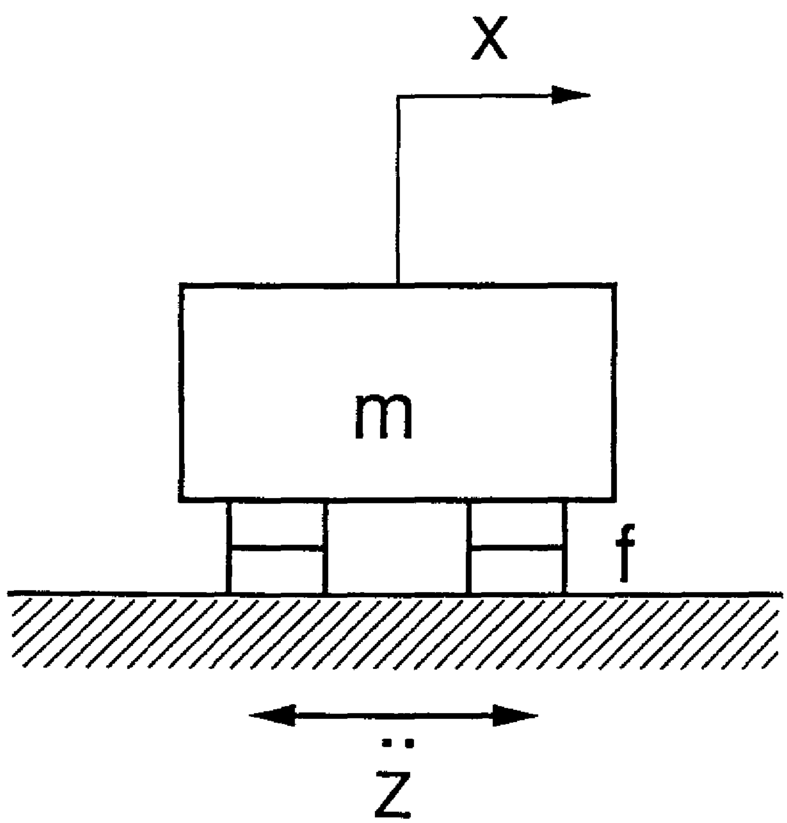

Figure 2.1: Analytical Model

In the sticking phase, Eq. 2.1 governs the motion of the structure until the Eq. 2.3 becomes true. As soon as the condition in Eq. 2.3 is met, the sliding phase starts and Eq. 2.2 governs the sliding motion. During the sliding phase, whenever $\dot{x}$ becomes zero, the criterion, Eq. 2.4, is checked to determine the subsequent behavior. Validity of the inequality given by Eq. 2.4 is the condition for entering the sticking phase [14]. That is, if the inequality holds, the structure will stick to the ground and Eq. 2.1 applies. If the criterion given by Eq. 2.4 is not satisfied, Eq. 2.1 will continue to govern the the subsequent sliding motion.

On the other hand, the normalized friction force f on the sliding interface between the structure and the ground is controlled by changing the fluid pressure in the bearing chamber through a pressure control system consisting of a computer, servo valve, amplifier, etc. The dynamic characteristics of the pressure control system are assumed to follow the first order time delay model:

$$T\dot{p} + p = u \tag{2.6}$$

where

p: pressure in fluid chamber of bearing

u: pressure control signal from computer

T: time constant

The normalized friction force f is negatively proportional to the pressure p in the bearing chamber:

$$f = -c_1 p + c_2 \tag{2.7}$$

where c_1 and c_2 are constants.

2.2 Bang-Bang Control

Bang-bang control approach provides a simple and yet often effective algorithm. The particular algorithm used in this study facilitates the following control: when the sliding displacement and the velocity of the mass are in the same direction, the pressure control signal $u(t)$ will be decreased to a minimum value u_{min} to increase the friction force in order to put the brake on the sliding. On the other hand, when the sliding displacement and the velocity are in the opposite direction, the pressure control signal will be increased to a maximum value u_{max} to decrease the friction force in order to make the sliding easier as much as possible;

$$u(t) = \begin{cases} u_{max}, & \text{if } \mathrm{sgn}(x) = -\mathrm{sgn}(\dot{x}) \\ u_{min}, & \text{if } \mathrm{sgn}(x) = \mathrm{sgn}(\dot{x}) \end{cases} \tag{2.8}$$

in which the control parameter, u_{max} and u_{min}, should be determined carefully, since they directly influence the control performance.

u_{max} should be set at a level as large as possible in order to reduce the friction force to a minimum level, making the mass slide as much as possible. It is, however, limited by the maximum pressure which can be applied to the bearing chamber. This depends on the weight W of the structure supported by the bearing:

$$u_{max} < \frac{W}{S} \tag{2.9}$$

where S is the vertically projected area of the fluid chamber of bearing. u_{min} should be set at a small level in order to confine the sliding displacement within an acceptable range.

This bang-bang control algorithm is simple and easy to implement in real-time on-line control operations, since the control signal only switches between two values, and only the sliding displacement needs to be measured by a sensor and fed back to the control signal.

Notice that $\text{sgn}(\dot{x})$ can be obtained by the displacement signal and does not need to measure the velociy

However, this algorithm has the following difficulty: In order to confine the sliding displacement within an acceptable range, the smaller value of u_{min} is needed. The smaller u_{min}, however, will lead to the larger seismic response acceleration of the structure and thus degrade the isolation performance. This conflict can only be alleviated, it appears, by taking advantage of pertinent optimal control algorithms.

2.3 Instantaneous Optimal Control

2.3.1 Formulation

As mentioned earlier, control theory has not been well developed for such systems in which the control force has the nonlinearity, unique to the friction controllable sliding isolation device. For this reason, an optimal control algorithm for control of the nonlinear friction system is developed on the basis of the instantaneous optimal control theory originally proposed by Yang et al. [12, 13].

The optimal pressure control signal $u(t)$ is determined by minimizing the following time dependent objective function $J(t)$ at every time instant t for the entire duration of an earthquake.

$$J(t) = q_d x^2(t) + q_f f^2(t) + r u^2(t) \tag{2.10}$$

in which the normalized friction force f equivalently represents the amount of response acceleration and also serves as a measure of the transfer of seismic force to the structure. The weighting coefficient q_d and q_f are non-negative and r is positive. They indicate the relative importance in the control objectives of the sliding displacement, response acceleration and pressure control signal, respectively. The basic objectives of the control is to make the structure slide as much as possible within an acceptable range and at the same time to ensure the transfer of seismic force to a minimum.

The following control algorithm is derived under the assumption that the structural motion is always in the sliding phase. The equation of motion given by Eq. 2.2 should be used as a constraint when minimizing the objective function $J(t)$. The first order time delay relationship between the control signal and the pressure described in Eq. 2.6, as well as the linear relationship between the friction and the pressure shown in Eq. 2.7, are also constraints. In the present formulation, however, these equations will be solved numerically using the Newmark's β method with $\beta = 1/6$ as shown below and these numerical solutions will be used as constraints:

$$\Delta\dot{x}(t) = \ddot{x}(t)\ \Delta t + \Delta\ddot{x}(t)\ \frac{\Delta t}{2} \tag{2.11}$$

$$\Delta x(t) = \dot{x}(t)\ \Delta t + \ddot{x}(t)\ \frac{\Delta t^2}{2} + \Delta\ddot{x}(t)\ \frac{\Delta t^2}{6} \tag{2.12}$$

$$\Delta f(t) = \dot{f}(t)\ \Delta t + \Delta\dot{f}(t)\ \frac{\Delta t}{2} \tag{2.13}$$

From the above equations, one obtains:

$$x(t) = x(t-\Delta t) + \dot{x}(t-\Delta t)\ \Delta t + \ddot{x}(t-\Delta t)\frac{\Delta t^2}{2} + [\ddot{x}(t) - \ddot{x}(t-\Delta t)]\frac{\Delta t^2}{6} \tag{2.14}$$

$$f(t) = f(t-\Delta t) + \dot{f}(t-\Delta t)\ \Delta t + [\dot{f}(t) - \dot{f}(t-\Delta t)]\frac{\Delta t}{2} \tag{2.15}$$

Furthermore,

$$x(t) = a\ f(t)\ \mathrm{sgn}(\dot{x}(t)) + b\ddot{z}(t) + d_1\ (t-\Delta t) \tag{2.16}$$

$$f(t) = -c\ u(t) + d_2(t-\Delta t) \tag{2.17}$$

where

$$a = b = -\frac{\Delta t^2}{6}, \qquad c = \frac{c_1 \Delta t}{2\,T + \Delta t} \tag{2.18}$$

$$d_1(t-\Delta t) = x(t-\Delta t) + \dot{x}(t-\Delta t)\Delta t + \frac{1}{3}\ \ddot{x}\ (t-\Delta t)\ \Delta t^2 \tag{2.19}$$

$$d_2(t-\Delta t) = \frac{2\,T}{2\,T+\Delta t}(\frac{c_2\ \Delta t}{2T} + f(t-\Delta t) + \frac{1}{2}\dot{f}(t-\Delta t)\Delta t) \tag{2.20}$$

The numerical solution to Eq. 2.2 given in Eq. 2.16, and the numerical solution to Eqs. 2.6 and 2.7 expressed in Eq. 2.17 are used as constraints as mentioned above. Thus, the following generalized objective function is established by introducing the Lagrangian multipliers λ_1 and λ_2:

$$\begin{aligned} H(t) &= q_d\, x^2(t) + q_f\, f^2(t) + r\, u^2(t) \\ &+\lambda_1 \left[x(t) - a\, f(t)\, \mathrm{sgn}(\dot{x}(t)) - b\, \ddot{z}(t) - d_1\, (t - \Delta t)\right] \\ &+\lambda_2 \left[f(t) + c\, u(t) - d_2\, (t - \Delta t)\right] \end{aligned} \tag{2.21}$$

The necessary conditions for minimizing the objective function $J(t)$ are:

$$\frac{\partial H}{\partial x} = 0, \quad \frac{\partial H}{\partial f} = 0, \quad \frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial \lambda_1} = 0, \quad \frac{\partial H}{\partial \lambda_2} = 0, \tag{2.22}$$

Substituting Eq. 2.21 into Eq. 2.22 yields the optimal pressure control signal:

$$u(t) = F_f\, f(t) + F_d\, x(t)\, \mathrm{sgn}(\dot{x}(t)) \tag{2.23}$$

where, the control feedback gains F_f and F_d are calculated by

$$F_f = \frac{c\, q_f}{r}, \quad F_d = \frac{c\, a\, q_d}{r} \tag{2.24}$$

and the displacement $x(t)$ and friction $f(t)$ are used for feedback purpose. Again, notice that $\mathrm{sgn}(\dot{x})$ can be obtained from the displacement signal without the need to measure the velocity. The friction is difficult to measure by a sensor, but the signal from the acceleration sensor $(\ddot{x}(t) + \ddot{z}(t))$ can be used instead of f in the SDOF structure. Therefore, the control signal becomes:

$$u(t) = F_f\, |\ddot{x}(t) + \ddot{z}(t)| + F_d\, x(t)\, \mathrm{sgn}(\dot{x}(t)) \tag{2.25}$$

In the development of the optimal control algorithm shown above, the time delay of the control device shown in Eq. 2.6 has been incorporated. In this case, the control is referred to as "instantaneous optimal control with time delay".

If the response of the control device is so fast that the time delay can be ignored, the relationship between the pressure and the control signal is given by:

$$p(t) = u(t) \tag{2.26}$$

The control algorithm is also developed under this condition, in which case the objective function and Hamiltonian become respectively:

$$J(t) = q_d x^2(t) + q_f f^2(t) + r\ u^2(t) \tag{2.27}$$

and

$$\begin{aligned} H(t) \quad = \quad & q_d\ x^2(t) + q_f\ f^2(t) + r\ u^2(t) \\ & + \lambda_1 \left[x(t) - a\ f(t)\ \mathrm{sgn}(\dot{x}(t)) - b\ \ddot{z}(t) - d_1\ (t - \Delta t) \right] \\ & + \lambda_2 \left[f(t) + c_1\ u(t) - c_2 \right] \end{aligned} \tag{2.28}$$

The control based on Eq. 2.26 is referred to as "instantaneous optimal control without time delay". By letting the following partial derivatives equal to zero,

$$\frac{\partial H}{\partial x} = 0, \quad \frac{\partial H}{\partial f} = 0, \quad \frac{\partial H}{\partial u} = 0, \quad \frac{\partial H}{\partial \lambda_1} = 0, \quad \frac{\partial H}{\partial \lambda_2} = 0, \tag{2.29}$$

the following control signal is obtained:

$$u(t) = F + F_d\ x(t)\ \mathrm{sgn}(\dot{x}(t)) \tag{2.30}$$

where

$$F = \frac{c_1 c_2 q_f}{r + q_f {c_1}^2}, \quad F_d = \frac{c_1 a q_d}{r + q_f {c_1}^2} \tag{2.31}$$

In this case, only the sliding displacement $x(t)$ needs to be measured and fed back.

Such instantaneous optimal control algorithms are also applicable to deal with other types of nonlinearity, by establishing a time dependent objective function and using numerical solutions of nonlinear equations as constraints to minimize objective function.

2.3.2 Sufficient Condition For Optimal Control

In the original derivations for the instantaneous optimal control algorithms shown above, the optimal control signals $u(t)$ are obtained from the necessary conditions, such as Eq. 2.22 and Eq. 2.29. In fact, the derived optimal control signals $u(t)$ also satisfy the sufficient conditions of optimality as will be proved in the following.

The sufficient condition for the optimal solution shown in Eq. 2.23 is given by [15, 16]:

$$[\delta x \; \delta f \; \delta u] \begin{bmatrix} \frac{\partial^2 H}{\partial x^2} & \frac{\partial^2 H}{\partial x \partial f} & \frac{\partial^2 H}{\partial x \partial u} \\ \frac{\partial^2 H}{\partial f \partial x} & \frac{\partial^2 H}{\partial f^2} & \frac{\partial^2 H}{\partial f \partial u} \\ \frac{\partial^2 H}{\partial u \partial x} & \frac{\partial^2 H}{\partial u \partial f} & \frac{\partial^2 H}{\partial u^2} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta f \\ \delta u \end{bmatrix} > 0 \tag{2.32}$$

Taking derivatives of H in Eq. 2.21 obtains:

$$\begin{gathered} \frac{\partial^2 H}{\partial x^2} = 2q_d, \qquad \frac{\partial^2 H}{\partial f^2} = 2q_f, \qquad \frac{\partial^2 H}{\partial u^2} = 2r, \\ \frac{\partial^2 H}{\partial x \partial f} = \frac{\partial^2 H}{\partial f \partial x} = 0, \quad \frac{\partial^2 H}{\partial x \partial u} = \frac{\partial^2 H}{\partial u \partial x} = 0, \quad \frac{\partial^2 H}{\partial f \partial u} = \frac{\partial^2 H}{\partial u \partial f} = 0 \end{gathered} \tag{2.33}$$

Substitution of Eq. 2.33 into the left hand side of Eq. 2.32 leads to the following expression:

$$2 \; [\; q_d \; (\partial x)^2 + q_f \; (\partial f)^2 + r \; (\partial u)^2 \;] > 0 \tag{2.34}$$

which is true since q_d and q_f are non-negative and r is positive, Eq. 2.32 is greater than zero. Thus, the sufficient condition for the optimal solution, Eq. 2.32, is satisfied.

Similarly, it can be shown that the optimal signal derived under the condition of no time delay, Eq. 2.30, also satisfies the sufficient condition.